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Jackson Integral Representations for solutions of quantized KZ equations

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The KZ equ is a fundumental differential equ of CFT with rich mathematical structures The KZ equ connects representation theories of Lie algebras and quantum groups. Connections come through integral representations for solutions of the KZ equ in terms of general hypergeometric functions.

Recently the KZ equ was quantized. The quntized KZ equ is a system of difference equations. It is expected that the qKZ equ will also connect two representation theories. The first is presumably the theory of representations of quantum groups. The second is the theory of yet undefined structure that may be called "a double quantum group". One may expect that connections between the two representation theories would come through integral representations for solutions of qKZ equ.

1. KZ equ.

Let g be a simple Lie algebra, $\Omega \in g \otimes g$ the tensor corresponding to an invariant scalar product, $V_1, \dots, V_n$ representations, $V = V_1 \otimes \dots \otimes V_n$. Let Ω_{ij} be the linear operator on V acting as Ω on $V_i \otimes V_j$ and as identity on other factors.

The KZ equ on an V-valued function $I(z_1, \dots, z_n)$ is the system

$$(1) \quad \frac{\partial I}{\partial z_i} = \frac{1}{\varkappa} \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} I \quad , \quad i = 1, \dots, n.$$

Here $\varkappa$ is a parameter of the equ.

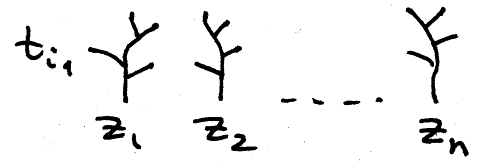
2. Integral representations for solutions, Hypergeometric functions.

There is a geometric sourse of differential equs of type (1). These are diff. equs for general hypergeometric functions.

I'll give a construction of such diff. equs.

Fix n, k. Consider letters $z_1, \dots, z_n, t_1, \dots, t_k$

An (n,k) forest $F = (T_1, \ldots, T_n)$

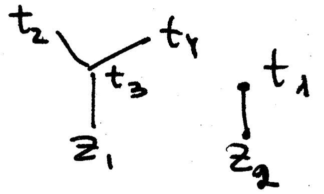


is a collection of n trees s.t.

1) # vertices $= n+k$ and hence # edges $= k$

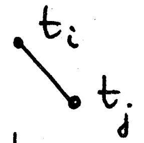
2) the root of T_j is marked by z_j, all other vertices are marked by $t_1, \ldots, t_k$.

Ex

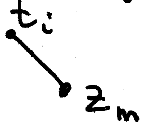


$(2,4)$ forest

Define diff form of an edge:



$\longmapsto \omega_{edge} = d(t_i - t_j)/(t_i - t_j)$



$\longmapsto d(t_i - z_m)/(t_i - z_m)$

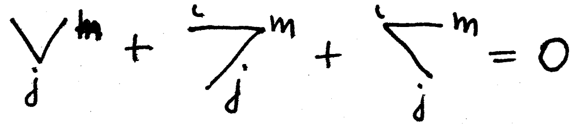
Define diff form of a forest: $\omega_{Forest} := \bigwedge_{edges} \omega_{edge}$

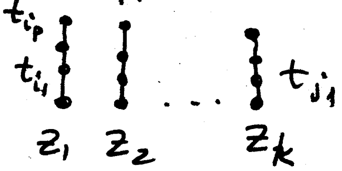
This is a k-form.

Relations among $\{\omega_F\}_F$ are based on the identity: set $\omega_{ij} = d(t_i - t_j)/(t_i - t_j)$ then

$$\omega_{ij} \wedge \omega_{jm} + \omega_{jm} \wedge \omega_{mi} + \omega_{mi} \wedge \omega_{ij} = 0.$$

Graphically it means:



<u>Basis</u> is formed by the differential forms of forests of trunks: 

Such a forest also may be denoted by

$$t_{i_p}\dots t_{i_1} z_1 \otimes \dots \otimes \dots\dots t_{j_1} z_m$$

This notation indicates a connection of the family of differential forms with tensor product of highest weight representations.

Consider a multivalued function

$$\ell(t,z) = \prod_{i<j}(z_i-z_j)^{a_{ij}/\varkappa}\prod_{i,m}(t_i-z_m)^{b_{im}/\varkappa}\times \prod_{i<j}(t_i-t_j)^{c_{ij}/\varkappa}$$

where $a, b, c, \varkappa$ are some complex parameters.

Fix $z_1,\dots,z_n$. In t-space fix a k-dim cycle γ, s.t.

1) γ lies outside singularities of ℓ,
2) a univalued branch of ℓ may be chosen over γ.

Consider the vector

$$I(z) = \left\{ \int_\gamma \ell(t,z)\,\omega_F \right\}_F$$

If $z_1,\dots,z_n$ are slightly deformed, then γ still satisfies 1) and 2), Hence $I(z)$ is a holomorphic function of z. It turns out that $I(z)$ satisfies a diff equ of the form (2)

$$(2)\quad \frac{\partial}{\partial z_i} I(z) = \frac{1}{\varkappa} \sum_{j \neq i} \frac{\Omega_{ij}(a,b,c)}{z_i - z_j} I(z)$$

<u>A claim</u> 1. For a given KZ eqn there exist a, b, c, k, s.t. equ (1) coincides with equ (2).

2. For a given equ (2) there exist a Kac-Moody Lie algebra $\mathfrak{g}$ and its repns $V_1, \dots, V_n$ s.t. the KZ equ (1) coincides with equ (2), see [1].

Changing γ we get another solution for (2). Solutions are parametrized by cycles. There are natural quantum group structures in the space of cycles. The de Rham duality between diff forms and cycles may be interpreted as a connection between Lie algebras structures and quantum group structures, see [2]

3. quantized KZ eqn.

Recently difference analogs of the KZ eqn were found: Smirnov (eqns for form factors), I. Frenkel, Reshetikhin (equs for matrix coeffic. of intertwining operators), Idzumi, Iohara, Jimbo, Miwa, Nakashima, Tokihiro, Nakayashiki (equs for correlation fns of the six-vertex model).

Fr.-Resh version.

Let $V_1, \dots, V_n$ be vector spaces, $V = V_1 \otimes \dots \otimes V_n$

Let $R_{V_i V_j} : V_i \otimes V_j \circlearrowleft$ be linear operators satisfying

QYB: $R_{ij}(x) R_{ik}(xy) R_{jk}(y) = R_{jk}(y) R_{ik}(xy) R_{ij}(x)$

Fix $p \in \mathbb{C}$.

The quntized KZ eqn on a V-valued function $I(z_1, \dots, z_n)$ is the system of difference equations

$$I(z_1, \dots, p z_i, \dots, z_n) = R_{V_i V_{i-1}}\left(\frac{p z_i}{z_{i-1}}\right) \dots R_{V_i V_1}\left(\frac{p z_i}{z_1}\right) \times$$

$$R^{-1}_{V_n V_i}\left(\frac{z_n}{z_i}\right) \dots R^{-1}_{V_{i+1} V_i}\left(\frac{z_{i+1}}{z_i}\right) I(z_1, \dots, z_n)$$

for $i = 1, \dots, n$.

Example. Let $V_1, \dots, V_n$ be finite dim. highest weight representations of $U_q sl_2$. Then there is a trigonometric R-matrix $R_{V_i V_j}(x)$ which gives a qKZ.

If $q = p^{\nu}$, $\nu \in \mathbb{C}$, and $q \to 1$ then $qKZ \to KZ$

I'll propos a geometric way to construct q KZ eqns. We'll start with a vector fn

$$I(z) = \{\int_\gamma \ell\, \omega_F\}_F$$

satisfying a diff. eqn $\frac{\partial I}{\partial z_i} = \frac{1}{\varkappa} \sum \frac{\Omega_{ij}}{z_i - z_j} I$.

We'll quantize this vector function in such a way that the quantization will satisfy a q-KZ. We'll replace $\int, \gamma, \ell, \omega_F$ by their discrete analogs.

IV <u>Jackson integrals</u>.

For $\xi = (\xi_1, \dots, \xi_k) \in \mathbb{C}^k$, the k-dim <u>p-cycle</u> $[0, \xi\infty]_p$ is the set $\{(p^{a_1}\xi_1, \dots, p^{a_k}\xi_k),\ a_1, \dots, a_k \in \mathbb{Z}\}$

The <u>Jackson integral</u> of a function $f(t_1, \dots, t_k)$ over a p-cycle $[0, \xi\infty]_p$ is the number

$$\int_{[0,\xi\infty]_p} f(t_1, \dots, t_k) \frac{d_p t_1}{t_1} \wedge \dots \wedge \frac{d_p t_k}{t_k} = (1-p)^k \sum_{a \in \mathbb{Z}^k} f(p^{a_1}\xi_1, \dots, p^{a_k}\xi_k)$$

if it exists.

V <u>p-analog of ℓ</u>. Set $(t)_\infty = \prod_{n=0}^{\infty} (1 - p^n t)$

A p-analog of $(1-t)^{2a}$ is the fn

$$(p^{-a}t)_\infty / (p^a t)_\infty$$

this fn tends to $(1-t)^{2a}$ when $p \to 1$.

p-analog of $\ell(t, z)$ is the function

$$l_p(t,z) = t_1^{\alpha_1}\cdots t_k^{\alpha_k} \prod \frac{((b_{ij})^{-1} t_i/t_j)_\infty}{(b_{ij}\, t_i/t_j)_\infty} \prod \frac{((c_{ij})^{-1} t_i/z_j)_\infty}{(c_{ij}\, t_i/z_j)_\infty}$$

where α, b, c are given numbers.

VI p-analog of ω_F is the following function φ_F

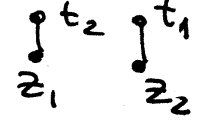
Let $F=(T_1,\dots,T_n)$ be a (n,k)-forest

Define the function of an edge $t_i \searrow t_j$ of a tree T_m as $\varphi_{edge} := \dfrac{z_m}{b_{ij}t_j - t_i}$,

of an edge $t_i \searrow z_m$ as $\varphi_{edge} := \dfrac{z_m}{c_{im} - t_i}$.

the function of a forest F as

$$\varphi_F = \Big(\prod_{edges} \varphi_{edge}\Big) \prod_{\substack{t_i\in T_\alpha \\ t_j \in T_\beta \\ i<j,\ \alpha>\beta}} \frac{t_j - b_{ij}t_i}{b_{ij}t_j - t_i} \prod_{\substack{i\in T_\alpha \\ l<\alpha}} \frac{z_l - c_{il}t_i}{c_{il}z_l - t_i}$$

Ex For $F =$ 

$$\varphi_F = \frac{z_1}{c_{21}z_1 - t_2}\,\frac{z_2}{c_{12}z_2 - t_1}\,\frac{z_1 - c_{11}t_1}{c_{11}z_1 - t_1}\,\frac{t_2 - b_{12}t_1}{b_{12}t_2 - t_1}$$

There are NO Relations among these fns.

VII Geometric qKZ. Fix $\xi \in \mathbb{C}^k$.

Claim The vector function

$$I_p(z) = \left\{ \int_{[0,\xi\infty]_p} l_p(t,z)\,\varphi_F(t,z)\, \frac{d_p t_1}{t_1}\wedge\dots\wedge\frac{d_p t_k}{t_k} \right\}_F$$

satisfies the qKZ eqn

$$I(\dots, pz_i, \dots) = R_{i,i-1}\left(\frac{pz_i}{z_{i-1}}\right) \dots R_{i,1}\left(\frac{pz_i}{z_1}\right) \times$$

$$R^{-1}_{n,i}\left(\frac{z_n}{z_i}\right) \dots R^{-1}_{i+1,i}\left(\frac{z_{i+1}}{z_i}\right) I(z_1, \dots, z_n)$$

for $i = 1, \dots, n$, where R_{ij} is defined as follows. (Precise statement see in [3])

VIII Solutions for QYB.

Forest come with theirown R-matrix.

Let $F = (T_1, T_2)$ be a $(2,k)$-forest. Set

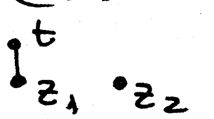
$$\overline{\varphi}_F(t,z) = \left(\prod_{\text{edges}} \varphi_{\text{edge}}\right) \prod_{\substack{t_i \in T_\alpha \\ t_j \in T_\beta \\ i<j,\ \alpha<\beta}} \frac{t_j - b_{ij} t_i}{b_{ij} t_j - t_i} \prod_{\substack{i \in T_\alpha \\ \ell > \alpha}} \frac{z_\ell - c_{i\ell} t_i}{c_{i\ell} z_\ell - t_i}$$

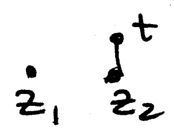
Claim. 1. $\overline{\varphi}_{T_1 T_2}(t,z) = \sum_{(T_1', T_2')} R^{T_1' T_2'}_{T_1 T_2}(z_1, z_2)\, \varphi_{T_1' T_2'}(t,z)$.

for some rational functions $R(z)$.

2. R depens only on z_1/z_2 and gives a unitary solution for the QYB, see [3].

IX Example. (2,1) Forests.

There are two (2,1) forests: 

and 

. Denote them by $f v_1 \otimes v_2$ and $v_1 \otimes f v_2$, respectively. Then

$$\varphi_{fv_1\otimes v_2} = \frac{z_1}{c_1 z_1 - t}, \quad \varphi_{v_1\otimes f v_2} = \frac{z_1 - c_1 t}{c_1 z_1 - t}\,\frac{z_2}{c_2 z_2 - t}$$

$$\overline{\varphi}_{fv_1\otimes v_2} = \frac{z_1}{c_1 z_1 - t}\,\frac{z_2 - c_2 t}{c_2 z_2 - t}, \quad \overline{\varphi}_{v_1\otimes f v_2} = \frac{z_2}{c_2 z_2 - t}$$

R-matrix:

$$\overline{\varphi}_{fv_1\otimes v_2} = \frac{c_1 z_2 - c_2 z_1}{c_1 c_2 z_2 - z_1}\,\varphi_{fv_1\otimes v_2} + \frac{((c_2)^2 - 1) z_1}{c_1 c_2 z_2 - z_1}\,\varphi_{v_1\otimes f v_2}$$

$$\overline{\varphi}_{v_1\otimes f v_2} = \frac{((c_1)^2 - 1) z_2}{c_1 c_2 z_2 - z_1}\,\varphi_{fv_1\otimes v_2} + \frac{c_2 z_2 - c_1 z_1}{c_1 c_2 z_2 - z_1}\,\varphi_{v_1\otimes f v_2}$$

X Concluding remarks. We have constructed a family of qKZ equs depending on parameters n, k, b, c. In the Field Theory qKZ equs depend on $\mathfrak{g}$, representations $V_1, \dots, V_n, \dots$

Open problem. Identify qKZ equs in the FT and qKZ equs for p-hypergeometric functions

The problem is solved for $U_q sl_2$ case (Matsuo [4] and Varchenko [3]). Namely special values for parameters of the hypergeometric construction were chosen in such a way that the corresponding qKZ equ gives the qKZ equ of the FT for $U_q sl_2$ case.

References.

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